PROSPECTIVE ELEMENTARY TEACHERS' MISUNDERSTANDINGS IN SOLVING RATIO AND PROPORTION PROBLEMS [1]

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This study explores difficulties that prospective elementary mathematics teachers have with the concepts of ratio and proportion, mainly when they are engaged in solving problems using algorithm procedures. These difficulties can be traced back to earlier experiences when they were students of junior and high school. The reflection on these difficulties by student teachers, comparing to informal ways of solving the problems is a fundamental step of the pre-service programme in which they are involved. In this communication I also present and discuss an attempt to promote development of prospective teachers' own knowledge of ratio and proportion as well as their awareness of the pupils' difficulties on this subject.

Introduction

Teacher education programmes should provide both a profound mathematics understanding of the basic concepts of mathematics and the capacity of future teachers to be aware of the difficulties and the misconceptions of the students (e.g. National Council of Teachers of Mathematics, 1989, 1991, 2001), in order to enable future teachers to create adequate learning situations for their students. One of the basic concepts of the Portuguese elementary mathematics curriculum (grades 6 -9) is the proportionality concept, which has been an obstacle to the learning of Mathematics. Most of the teachers teach this theme in a very formal way, emphasising the memorisation of the rules. Researchers on proportional reasoning have been looking for children's strategies and errors (e.g. Hart, 1984, Lamon, 1993, Vergnaud, 1988). Vergnaud has mainly distinguished two kinds of strategies to solve situations involving direct proportions: the scalar operator "within" the same magnitude and the functional (across the measures) "between" the two magnitudes. Teachers should be aware of their own strategies and errors when they are dealing with situations of proportionality both direct (isomorphism of measures) and "inverse" (product of measures). They need to understand the differences between these two structures, respecting the invariance and the Cartesian graphs and to know what is the model that is underlying a specific problem.

In this study I will focus on the procedures and on the reasoning of future teachers in solving proportionality problems in the context of a teacher education programme. I

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begin by providing a short explanation of the programme inasmuch as it seems important to understand the process that prospective teachers followed until being prepared to elaborate lessons for their future pupils to teach ratio and proportion.

The teacher education programme

The teacher education programme of this study lasts four years, has 28% dedicated to mathematics education, and during the two last years prospective teachers have 6 weeks in the third year and 3 months in the fourth year, of practice in real classrooms. During the practice they are responsible to plan lessons for their "borrowed" 5^{th} and 6^{th} grade pupils. They are accompanied by the teacher of the class and by the supervisor, that is a professor of the Higher School of Education.

What mathematics content elementary teachers need to know, and how their knowledge of mathematics relates to teaching practice and students' learning, are questions that every institution of teacher education should address. Literature in the field (e.g. Wood and Cobb, Yackel, 1991, and Ball, Lubienski, Mewborn, 2001) agree that mathematical content knowledge is important, but is not enough. The process by which teachers learn, the awareness of students' difficulties and the reflection upon their own understanding of mathematics are key features to take into account in pre-service courses. From early days until nowadays, many authors (e.g. Dewey, 1916, Erault, 1977, Zeichner, 1993) advocate the development of reflective thinking in teacher education. Reflection is focused as a very relevant feature to professional growth. One step to understanding the pupil's difficulties is to experience themselves these difficulties and reflect upon them (e.g. Zeichner, 1993). As the way how teachers learn influences the way how teachers work within their classrooms (e.g. National Council of Teachers of Mathematics, 1991), the programme intends to foster in prospective teachers the capacity to design problems, tasks and projects for their students which can provide a meaningfully understanding of mathematics. This understanding of mathematics should be deeper in prospective teachers, "it requires that teachers themselves also understand the central ideas of their subjects, see relationships, and so forth" (Ball, Lubienski and Mewborn, 2001, p.43).

Based on these ideas, the mathematics course of the teacher programme, in which these prospective teachers were engaged, underlies three main principles: *Experimentation, reflection,* and *transference*. Experimentation since teachers should experiment mathematical activities and not just listen to transmitted knowledge. Reflection provides thinking and discussion on several aspects: the consideration of their own thought process and of the others of solving the task, the mathematical knowledge and concepts that model it, and the students' difficulties. This process can develop a deep understanding of mathematics, mainly related to elementary mathematics education After that, future teachers are encouraged to product plans of lessons and materials to teach the pupils, in a process of *transferring.* Transference is very crucial, since prospective teachers have during the education programme to do their practice in real classrooms, which provides an evaluation and reflection of their planned lessons.

Ratio and proportion

In spite of the first explicit indication of ratio and proportion being in the 6th grade (11 years old children) curriculum, during the primary level (until 4th grade) Portuguese teachers teach proportion problems using the unit rate approach. For example, if 2 packs of cereals cost 10 euros, how much cost 4 packs? Primary teachers incite pupils to find the cost of one pack and then multiply by 4 to find the price of four packs. During the 6th grade teachers introduce the ratio notion mainly comparing the number of the elements of two discrete sets, as the number of girls and boys in a classroom. The proportion relationship is introduced as the comparison of two ratios, and students are asked to solve problems using equations such as $\frac{a}{b} ? \frac{c}{x}$ (the variable can have all the four positions) calculating the answer by the cross-product and divide algorithm, or the rule of three that has the same algorithm but that does not use fractions. For example:

packs	euros	
2	10	x= 4 x 10 : 2
4	x	

It is interesting to note that in 5th and 6th grades textbooks (and probably most of the teachers, as they followed the manual) the reducing to the unit strategy as described above is not used. Also, for instance, in $\frac{5}{15} ? \frac{x}{45}$, students are nor encouraged to look for the x, by multiplying 3 by 5 using the equivalence of fractions, but they are taught to use the rule.

A considerable amount of research has been developed approaching ratio and proportion, as well as investigating children errors and strategies when attempting to solve problems in this area (Hart, 1984, 1988, Vergnaud, 1988, Kieren,1988, Cramer, Post and Currier, 1993, Behr, Khoury, Harel, Post, and Lesh, 1997). One source of difficulties may be a consequence of proportional reasoning to be a form of mathematical reasoning that involves a sense of co-variation and multiple comparisons (Cramer, Post and Behr, 1988). These authors refer Piaget stating that "the essential characteristic of proportional reasoning is that it must involve a **relationship between two relationships** (i. e. a "second-order"relationship)" (p. 94). Some researchers, (e.g. Cramer, Post and Currier, 1993) call the attention to the fact that someone can solve a proportion using the algorithm, but this does not necessarily mean that he or she are mobilising proportional reasoning. Vergnaud analyses proportion situations in the conceptual field of multiplicative structures as they involve multiplication and division, and he says that "difficulties which students may be due to the context of application more than to multiplicative structures themselves" (p. 142).

Methodology

Participants: 19 pre-service teachers from a public High School of Education. They are attending a pre-service teacher programme to be teachers of 5^{th} and 6^{th} grades of mathematics and sciences; They are in the 4^{th} year, the last one of the course. All of them completed the secondary school studying Mathematics. When they begin the programme they have a view of mathematics mostly as computation and rules; they have few autonomy, expecting that the teacher explains and afterwards they practice. During their teacher education programme, mathematics and its teaching and learning is based on problem solving and manipulative activities, following the process of the teacher education programme described above.

Prospective teachers tasks: *First task* - Participants were asked to solve three kind of problems, one simple direct proportional situation, one situation with an additive relation between the variables and an inverse proportional situation (product of measures) which are modelling by Y= kx, y=x+k, and Y=k/x, respectively. All of the situations have a constant K, but just the first and the third are situations in the conceptual field of multiplicative structures (Vergnaud, 1983 , 1988). There was no discussion after this task, which was followed by the second one.

Second task: Participants had to solve four problems, one additive relation, two simple direct proportional problem, one with a missing value and the other involving a numerical comparison of two rates. The fourth problem is a product of measures situation. They are told to solve the problems but now without using any known algorithm and to write an explanation of their reasoning in each problem. After that, prospective teachers discussed in a plenary session the processes they found to solve the problems.

According to the teacher education programme these two tasks are included in the experimentation phase. The following phase provided a long reflection upon the way they solved the problems in both tasks. Teachers were invited to find the similarities and the differences among the problems and to find the mathematical relation between the variables as well as to trace the Cartesian graphs of the situations. Students were asked to find new situations of the three mathematical models. Discussions took place in small groups of 4/5 students and in plenary sessions.

After that a more detailed attention was given to the direct proportion problems as they relate to the children curricula. In addition both scalar relationship and the function relationship were analyzed, as well as the "building up" strategy often used by children. Also problems of missing value, of comparison when the four values are known, use of whole numbers, fractional and decimals were discussed in a perspective of children learning.

Finally students were asked to develop plans of lessons to the 6th grade students. To develop these lessons they had textbooks used in schools (in Portugal there is not a unique manual), books and journals of mathematics education.

Data collection: The data was collected by the answers of the two tasks that were carried out individually, notes of the discussions in groups and plenary sessions, and the lessons that they prepared for children and that were developed in groups of 4/5 students.

Main findings and discussion of prospective teachers' errors and strategies

First Task: The following table resumes the errors and the strategies followed by students.

Table 1. Percentages of right and wrong answers and Strategies used (N=19)

Problems	**Right solution**	**Wrong solution**	**Strategy used for right solution**	**Strategy used for wrong solution**
Additive relation **Y= K+X**	42%	58%	They did a schema or they calculate	100% of the mistakes are due to use of the rule of three
Direct proportion (missing value) **Y= K X**	100%	______	74% calculate the price of the unit and then multiply. 26%used the rule of three	______
Inverse proportion **Y= K/X**	58%	42%	They used a scalar Factor, multiply one variable by the scalar and divide the other by the same scalar	75% used the rule of three 25% used a wrong schema

***Second Task*:** The following table resumes the errors and the strategies followed by students. Remember that in this task the prospective teachers are asked not to use any rule to solve the problems and they were also asked to explain the procedure.

Table 2. Percentages of right and wrong answers and Strategies used (N=19)

Problems	**Right solution**	**Wrong solution**	**Strategy used for right solution**	**Strategy used for wrong solution**
Additive relation	100%	______	They did a schema	
Direct proportion Missing value task ***(Mr Short and Mr Tall problem, Hart, 1984)***	74%	26%	50% use the unit rate strategy, 50% use the scalar ("within") strategy	100% of mistakes are due to an additive reasoning
Direct proportion Comparison task (4 quantities are known) ***"In a table 10 children share 6 pizzas, in another table 8 children share 5 pizzas. In what table each children eats more pizza?"***	68%	32%	46% calculate the amount of pizza for each children and then compare the ratios. 54%used an iconic representation.	67% of mistakes are due to an error of calculation of the ratio 33% of mistakes are due to an additive reasoning

Inverse proportion ***12 workers open a trench, in 5 days, how many days did 3 workers need, if they did the same work in the same time.***	79%	21%	All used a scalar Factor, multiply one variable by the scalar and divide the other by the same scalar	75% of the mistakes are due to an incorrect use of the scalar factor, they used it as a direct proportion 25% use the additive reasoning

In first task, *only seven prospective teachers did all the three problems in a right way*. Two students always used the rule of three in all the situations, and another two made the mistake of using the rule of three in both first and third problem but they did not use it in the second, when it was indeed the right procedure, preferring the unit rate strategy instead. These findings seem to show that when students try to solve a problem with three known values and a fourth one unknown (a missing value problem), they choose the rule of three. During the discussion of the students strategies, most of them stated that in junior and secondary school they always followed the procedure of the rule of three. "Even in Physics I remember using this rule to solve a lot of problems" stated one student. The second task followed immediately the first task, without any discussion in the class. In this task a greater number of students gave a right answer, however five students (26%) used an additive reasoning in the missing value problem of direct proportionality. *Ten students solved all the four problems in a right way, and one student did all of them in a wrong way.* They used more schemas now than in the first task. It is interesting to note that in the direct proportion problem of the first task everybody had the right solution (most of them using the unit rate), and in the second task 32% failed. Perhaps this is due to the kind of problem (the first asked to know the price of 24 balloons knowing the price of three), the second asked to know how many paperclips were needed for Mr. Tall's height, knowing both Mr. Short's height in paperclips and matchsticks and Mr. Tall's height in matchsticks. During the group and plenary discussion of the *reflection phase* the students referred that they never did proportional problems without using a rule. They also stated that when they studied functions in the secondary school, they had studied the function Y= K/X as well as the linear function, but always without perceived that they could be related to concrete problems like those of these tasks.

The analysis of the plans of lessons which they developed in the context of the *transferring phase* of the course, shows that prospective teachers were very aware of the possible errors of their future pupils. Most of them chose to begin by letting them to solve problems by using informal strategies, after exploring the scalar factor within the variables and the unit rate. They also created tasks to compare situations of proportionality and others, and at last the comparison between two quantities of a same discrete set. As the official curriculum explicitly focus the algorithms and rules they thought that they should teach also these aspects. So they dedicated one lesson to the rules and practice exercises.

General comments of the course: the strategy followed during this course has provided knowledge about some misunderstandings that prospective elementary mathematics teachers have with ratio and proportion concepts. The course had as a start point the self-awareness of these misunderstandings and a shared reflection about them, as well as the study of the mathematics content underlying these subjects. The elaboration of the lessons will be very useful during the pedagogical practice in real classroom inasmuch as it will provide an evaluation and a reflection about the lessons implementation, which will be developed later on.

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